Exercise Set #5

Exercise 1: Given an integer s, find a link descriptions of $\Sigma(2, 2s-1, 2s+1)$.

Exercise 2: Let Σ be the Brieskorn homology sphere $\Sigma(2, 7, 13)$.

a) Prove that $\mu(\Sigma) = 0$.

b) Use Furuta's theorem to prove that Σ is not homology cobordant to zero.

c) Prove that Σ has an infinite order in the homology cobordism group.

Exercise 3: A knot K in $S^3 = \partial D^4$ is *slice* if there is a disk $D \subset D^4$ such that $\partial D = K$. Prove that, for any integer n, the integral homology sphere obtained by $(1/n)$-surgery on a slice knot in S^3 is homology cobordant to zero.

Exercise 4: Prove that if a homology 3-sphere Σ can be embedded in $\mathbb{R}^4$ then $\mu(\Sigma) = 0$. In particular, the Poincaré homology sphere cannot be embedded in $\mathbb{R}^4$.

Exercise 5: Prove that any Seifert homology sphere $\Sigma(a_1 \ldots, a_n)$ with even a_1 can be obtained by a surgery according to an even-weighted star-shaped graph.

Exercise 6: Let $K_1 \subset \Sigma_1$ and $K_2 \subset \Sigma_2$ be oriented knots in oriented homology spheres, M_{K_1} and M_{K_2} their exteriors, and (m_1, l_1) and (m_2, l_2) the canonical meridian-longitude pairs on ∂M_{K_1} and ∂M_{K_2}, respectively. By the *splice* of Σ_1 and Σ_2 along K_1 and K_2 we will mean the manifold $\Sigma = M_{K_1} \cup M_{K_2}$ obtained by gluing M_{K_1} and M_{K_2} along their boundaries by an orientation reversing homeomorphism matching m_1 to l_2 and l_1 to m_2.

a) Prove that Σ is a homology sphere.

b) Define a trivial knot in a homology sphere Σ as an unknot in a copy of $D^3 \subset \Sigma$. Prove that the splice of Σ_1 and Σ_2 along trivial knots is $\Sigma_1 \# \Sigma_2$.

c) Let homology spheres Σ_1' and Σ_2' be obtained from Σ_1 and Σ_2 by (-1)-surgery on, respectively, K_1 and K_2. Let $K_1^* \subset \Sigma_1'$ and $K_2^* \subset \Sigma_2'$ be the images of the canonical longitudes l_1 and l_2. Prove that the splice of Σ_1 and Σ_2 along K_1 and K_2 is homeomorphic to the homology sphere obtained from $\Sigma_1' \# \Sigma_2'$ by $(+1)$-surgery on the knot $K_1^* \# K_2^*$.

Exercise 7: Let Σ be the splice of homology spheres Σ_1 and Σ_2 along knots K_1 and K_2. Prove that $\mu(\Sigma) = \mu(\Sigma_1) + \mu(\Sigma_2)$.