Heegaard Floer homology and Dehn surgery

Problem Set 1

Problem 1. Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be n mutually disjoint, simple closed curves on a closed oriented surface Σ. Prove that the homology classes $[\alpha_1], \dots, [\alpha_n] \in H_1(\Sigma)$ are linearly independent if and only if the complement $\Sigma \setminus (\alpha_1 \cup \dots \cup \alpha_n)$ is connected.

Problem 2. Find a genus 1 Heegaard diagram of S^3, and use it to compute $HF^\infty(S^3), HF^-(S^3), HF^+(S^3)$.

Problem 3. Let

$$\big(\Sigma, \{\alpha_1, \dots, \alpha_g\}, \{\beta_1, \dots, \beta_g\}\big)$$

be a Heegaard diagram of Y. Prove

$$H_1(Y) \cong H_1(\Sigma)/\langle [\alpha_1], \dots, [\alpha_g], [\beta_1], \dots, [\beta_g] \rangle.$$

Problem 4. Prove the map $\delta\colon \ \mathrm{Spin}^c(Y) \to H^2(Y)$ is a one-to-one correspondence.

Problem 5. Suppose $\mathfrak{s}_1, \mathfrak{s}_2 \in \mathrm{Spin}^c(Y)$, prove

$$\delta(\mathfrak{s}_1, \mathfrak{s}_2) = +\delta(\overline{\mathfrak{s}_2}, \overline{\mathfrak{s}_1}), \quad c_1(\mathfrak{s}_1) - c_1(\mathfrak{s}_2) = 2\delta(\mathfrak{s}_1, \mathfrak{s}_2).$$

As a consequence, show that the map $c_1\colon \mathrm{Spin}^c(Y) \to H^2(Y)$ is injective if $H_1(Y)$ has no 2–torsion.

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Problem Set 2

Problem 1. Prove that $\widehat{HF}(Y,\mathfrak{s}) \neq 0$ if and only if $HF^+(Y,\mathfrak{s}) \neq 0$.

Problem 2. Let Y be a rational homology sphere, $\mathfrak{s} \in \mathrm{Spin}^c(Y)$. Then the following conditions are equivalent:
(1) $\widehat{HF}(Y,\mathfrak{s}) \cong \mathbb{Z}$,
(2) $HF^-(Y,\mathfrak{s}) \cong \mathbb{Z}[U]$,
(3) $HF^+(Y,\mathfrak{s}) \cong \mathbb{Z}[U,U^{-1}]/U\mathbb{Z}[U]$,
(4) $HF_{\mathrm{red}}(Y,\mathfrak{s}) = 0$.

Problem 3. If $c_1(\mathfrak{s})$ is torsion, then the map $HF^\infty(Y,\mathfrak{s}) \to HF^+(Y,\mathfrak{s})$ is an isomorphism when the grading is sufficiently high, and the map $HF^-(Y,\mathfrak{s}) \to HF^\infty(Y,\mathfrak{s})$ is an isomorphism when the grading is sufficiently low.

Problem 4. Let Y be a closed oriented connected 3–manifold, $\mathfrak{s} \in \mathrm{Spin}^c(Y)$. Prove that $U\colon HF^\infty(Y,\mathfrak{s}) \to HF^\infty(Y,\mathfrak{s})$ is an isomorphism. In particular, if $c_1(\mathfrak{s})$ is torsion, show that there exists a finitely generated abelian group A, such that $HF^\infty(Y,\mathfrak{s})$ is isomorphic to $A[U,U^{-1}]$.

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Problem Set 3

Problem 1. Suppose that $K \subset S^3$, n is a positive integer. Prove that the set

$$\{\mathfrak{s} \in \mathrm{Spin}^c(S^3_K(n)) | \, HF_{\mathrm{red}}(S^3_K(n), \mathfrak{s}) \neq 0\} \; .$$

has at most $2g(K)-1$ elements. In particular, if Y is a rational homology sphere, and there are exactly N Spin^c structures $\mathfrak{s} \in \mathrm{Spin}^c(Y)$ satisfying $HF_{\mathrm{red}}(Y, \mathfrak{s}) \neq 0$, then Y cannot be obtained by integer surgery on any knot in S^3 with genus $\leq \frac{N}{2}$.

Problem 2. Let $K \subset S^3$ be an L-space knot, $C = CFK^\infty(S^3, K)$, $k \in \mathbb{Z}$.
(1) Prove that $H_*(C\{i < 0, j \geq k\}) \cong \mathbb{Z}\langle 1, U^{-1}, \dots, U^{1-t}\rangle$ for some integer $t \geq 0$.
(2) Prove

$$\chi(C\{i < 0, j \geq k\}) = t_k = \sum_{n=1}^{\infty} n a_{n+k},$$

where a_i's are the coefficients of the normalized Alexander polynomial.
(3) Prove $t = t_k$.

Problem 3. Let $K \subset S^3$ be an L-space knot, $C = CFK^\infty(S^3, K)$, $k \in \mathbb{Z}$.
(1) Prove that $H_*(C\{\max(i, j-k) = 0\}) \cong \mathbb{Z}$.
(2) Prove that $H_*(C\{i < 0, j = k\})$ is either 0 or $\mathbb{Z}$, the same is true for $H_*(C\{i = 0, j \leq k\})$.
(3) Prove that exactly one of the two groups $H_*(C\{i < 0, j = k\})$ and $H_*(C\{i = 0, j \leq k\})$ is $\mathbb{Z}$.
(4) Prove that if $H_*(C\{i = 0, j = k\}) \cong \mathbb{Z}^2$, then both $H_*(C\{i < 0, j = k\})$ and $H_*(C\{i \leq 0, j = k\})$ are $\mathbb{Z}$.
(5) Prove that $H_*(C\{i = 0, j = k\})$ is either 0 or $\mathbb{Z}$. As a consequence, the coefficients of the Alexander polynomial of an L-space knot are 0 or ± 1.